

Geometry II

Solutions to Coursework #4

$$1. a) K(t) = \frac{\|\dot{\gamma} \wedge \ddot{\gamma}\|}{\|\ddot{\gamma}\|^2} \quad (1), \quad T(t) = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} \quad (1)$$

if $\tilde{\gamma}(\tilde{t})$ is a reparametrization of $\gamma(t)$, so that ~~$\tilde{\gamma}(\tilde{t}) = \gamma(\tilde{\phi}(\tilde{t}))$~~ $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for some function ϕ then $K(\tilde{t}) = K(t)$ and $T(\tilde{t}) = T(t)$ (3)

$$b) \gamma(t) = (t, \cosh t, 0)$$

$$\dot{\gamma}(t) = (1, \sinh t, 0)$$

$$\|\dot{\gamma}\|^2 = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$$

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du = \int_0^t \cosh u du \\ = \sinh u \Big|_0^t = \sinh t$$

$$c) t = \sinh^{-1} s$$

$$\tilde{\gamma}(s) = (\sinh^{-1} s, \cosh(\sinh^{-1} s), 0)$$

$$= (\sinh^{-1} s, \sqrt{1 + \sinh^2(\sinh^{-1} s)}, 0)$$

$$= (\sinh^{-1} s, \sqrt{1 + s^2}, 0).$$

$$2, a) \gamma(t) = (t, t^2, t^3), \quad \dot{\gamma}(t) = (1, 2t, 3t^2), \quad \ddot{\gamma}(t) = (0, 2, 6t) \quad (2)$$

$$\ddot{\gamma}(t) = (0, 0, 6).$$

$$\dot{\gamma} \times \ddot{\gamma} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix} = (6t^2, -6t, 2) \quad (2)$$

$$\|\dot{\gamma}\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$K(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}} \quad (2)$$

$$\Gamma = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{(6t^2, -6t, -2) \cdot (0, 0, 6)}{4 + 36t + 36t^4} = \frac{3}{1 + 9t^2 + 9t^4} \quad (2)$$

$$\lim_{t \rightarrow \infty} T = 0. \quad (2)$$

$$b) \hat{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) = \gamma(e^{\tilde{t}}) = (e^{\tilde{t}}, e^{2\tilde{t}}, e^{3\tilde{t}}) \quad (1)$$

$$t = e^{\tilde{t}} \Rightarrow \tilde{t} = \log t \quad t \in (0, \infty) \Rightarrow \alpha = 0, \beta = \infty \quad (2)$$

$$\alpha = \lim_{t \rightarrow 0} \log t = -\infty, \quad \beta = \lim_{t \rightarrow \infty} \log t = \infty \quad (2)$$

$$c) \tilde{\gamma}' = (e^{\tilde{t}}, 2e^{2\tilde{t}}, 3e^{3\tilde{t}})$$

$$\tilde{\gamma}'' = (e^{\tilde{t}}, 4e^{2\tilde{t}}, 9e^{3\tilde{t}}) \quad (2)$$

$$\tilde{\gamma}''' = (e^{\tilde{t}}, 8e^{2\tilde{t}}, 27e^{3\tilde{t}})$$

$$\tilde{\gamma}' \times \tilde{\gamma}'' = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{\tilde{t}} & 2e^{2\tilde{t}} & 3e^{3\tilde{t}} \\ e^{\tilde{t}} & 4e^{2\tilde{t}} & 9e^{3\tilde{t}} \end{pmatrix} \quad (16)$$

$$= (6e^{5\tilde{t}}, -6e^{4\tilde{t}}, 2e^{3\tilde{t}}) \quad (1)$$

$$\|\tilde{\gamma}'\| = \sqrt{e^{2\tilde{t}} + 4e^{4\tilde{t}} + 9e^{6\tilde{t}}} \quad (1)$$

$$K(\tilde{t}) = \frac{\sqrt{36e^{10\tilde{t}} + 36e^{8\tilde{t}} + 4e^{6\tilde{t}}}}{(e^{2\tilde{t}} + 4e^{4\tilde{t}} + 9e^{6\tilde{t}})^{3/2}} \quad (2)$$

$$= \frac{e^{3\tilde{t}} \sqrt{36e^{4\tilde{t}} + 36e^{2\tilde{t}} + 4}}{e^{3\tilde{t}} (1 + 4e^{2\tilde{t}} + 9e^{4\tilde{t}})^{3/2}} \quad (1)$$

$$= K(\mathbf{t}) \text{ when } t \text{ is replaced by } e^{\tilde{t}}. \quad (2)$$

$$d) \quad \tau = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{1}{\sqrt{1+4t^2+9t^4}} (1, 2t, 3t^2) \quad (3)$$

$$\beta = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} = \frac{1}{\sqrt{36t^4 + 36t^2 + 4}} (6t^2, -6t, 2) \quad (3)$$

$$\gamma = \beta \times \tau = \frac{1}{\sqrt{(1+4t^2+9t^4)(36t^4+36t^2+4)}} \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 6t^2 & -6t & 2 \end{pmatrix} \quad (2)$$

$$= \frac{1}{\sqrt{(1+4t^2+9t^4)(36t^4+36t^2+4)}} (12t^3+6t, 2-18t^4, 12t^2+6t) \quad (2)$$

$$3. a) \quad \|\beta \times \tau\| = \|\beta\| \cdot \|\tau\| \sin \theta$$

$$= 1 \cdot 1 \cdot 1 = 1 \quad \text{as } \theta = \frac{\pi}{2}$$

(5)

$$b) \quad \tau \times \beta = (\beta \times \tau) \times \beta = (\beta \cdot \beta) \tau - (\beta \cdot \tau) \beta$$

$$= 1(\tau) - 0(\beta)$$

$$= \tau$$

(5)

$$\tau \times \tau = \tau \times (\beta \times \tau) = (\tau \cdot \tau) \beta - (\tau \cdot \beta) \tau$$

$$= 1(\beta) - 0(\tau) = \beta$$

(5)

$$4. a) \quad \gamma = \left(\frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right)$$

$$\dot{\gamma} = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right) \quad (2)$$

$$\ddot{\gamma} = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t \right) \quad (2)$$

~~BAW~~ ~~BAW~~ ~~BAW~~

$$\ddot{\gamma} = \left(\frac{4}{5} \sin t, \cos t, -\frac{3}{5} \sin t \right) \quad (2)$$

$$K = \|\ddot{\gamma}\| = \sqrt{\frac{16}{25} \cos^2 t + \sin^2 t + \frac{9}{25} \cos^2 t}$$

$$= \sqrt{\cos^2 t + \sin^2 t} = 1$$

~~BAW~~ ~~BAW~~ (4)

$$\Gamma = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

$$\dot{\gamma} \times \ddot{\gamma} = \det \begin{pmatrix} i & j & k \\ -\frac{4}{5} \sin t & -\cos t & \frac{3}{5} \sin t \\ -\frac{4}{5} \cos t & \sin t & \frac{3}{5} \cos t \end{pmatrix} = \left(-\frac{3}{5}, 0, -\frac{4}{5} \right)$$

~~BAW~~ ~~BAW~~ (4)

$$(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dot{\mathbf{r}} = -\frac{12}{25} \sin t + \frac{12}{25} \sin t = 0 \Rightarrow T=0. \quad (18) \quad (4)$$

$$b) \quad \tau = \dot{\mathbf{r}} = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t\right) \quad (4)$$

$$\mathbf{v} = \ddot{\mathbf{r}} = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t\right) \quad (4)$$

$$\beta = \tau \times \mathbf{v} = \dot{\mathbf{r}} \wedge \ddot{\mathbf{r}} = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right) \quad (4)$$

c) Since $T=0$, $\mathcal{H}$ is contained in a plane, (3)

A vector orthogonal to that plane

$$\text{is } \beta = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right).$$

(4) (4) (2)